Why the Pennington-Wilson expansion with real coefficients is of little use in the analysis of production processes

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Abstract

We critically analyse and comment on the claims of M. R. Pennington and D. J. Wilson [1]. Although we generally agree with their obvious algebra, it is clearly not applicable to our equations. Moreover, we argue that the corresponding proposal is not useful for production-data analysis. The advantages of our approach, which involves complex yet purely kinematical coefficients, are summarised.

The analysis of two-body subamplitudes in processes of strong decay is often carried out under the spectator assumption [2–4]. In such cases, one may express the two-body production subamplitude $\boldsymbol{P}$ as a linear combination of elements of the two-body scattering amplitude T [5–8]. The latter quantities, which contain the full two-body dynamics, are then supposed to be known, either from experiment [9–19], or from theoretical considerations [20–26].

In Ref. [27], assuming quark-pair creation within a non-relativistic framework, we deduced a relation between a subamplitude $\boldsymbol{P}$, describing a meson pair emerging from the products of a strong three-meson decay process, and the corresponding two-meson scattering amplitude T, reading[1]

$$\boldsymbol{P} \;=\; \Re e\left(\boldsymbol{Z}\right) \;+\; i\,T\,\boldsymbol{Z} \quad , \tag{1}$$

where $\boldsymbol{Z}$ consists of complex functions, smooth in the two-body total invariant mass, which are of a kinematical origin and do not carry information on the two-body interactions. So far, our result (1) is fully in line with the conclusions of Ref. [5], which was also based on the OZI rule [28]

[1] In Appendix A we give the precise relation between the expressions used in Ref. [27] and $\boldsymbol{Z}$.

and the spectator picture. In the latter paper, it was found that the production amplitude can be written as a linear combination of the elastic and inelastic two-body scattering amplitudes, with coefficients that do not carry any singularities, but are rather supposed to depend smoothly on the total CM energy of the system. However, our result (1) seems to be in conflict with yet an extra constraint on $\boldsymbol{Z}$, postulated in Ref. [5] and later also in Ref. [6], namely that the production amplitude should be given by a *real* linear combination of the elements of the transition matrix, owing to the unitarity relation

$$\Im m\,(\boldsymbol{P}) \;=\; T^{*}\,\boldsymbol{P}\ . \tag{2}$$

The latter property follows straightforwardly from the operator relations $\boldsymbol{P}V = (1+TG)V = V + TGV = T$, the symmetry of T, the realness of V, and the unitarity of $1+2iT$, which gives $\Im m\,(\boldsymbol{P})\,V = \Im m\,(\boldsymbol{P}V) = \Im m\,(T) = T^{*}T = T^{*}\boldsymbol{P}V$. This leads, for non-singular potentials V, to Eq. (2).

Now, in Ref. [29] we proved that our expression (1) also satisfies relation (2). However, in Ref. [1] M. R. Pennington and D. J. Wilson showed how from Eq. (1), with the definition

$$\boldsymbol{Q} \;=\; T^{-1}\,\Re e\,(\boldsymbol{Z}) \;+\; i\,\boldsymbol{Z}\quad , \tag{3}$$

one obtains

$$\boldsymbol{P} \;=\; T\,\boldsymbol{Q}\quad . \tag{4}$$

When one furthermore makes use of the property

$$\Im m\left(T^{-1}\right) \;=\; -\,\mathbf{1}\quad , \tag{5}$$

one finds from Eq. (3)

$$\Im m\,(\boldsymbol{Q}) \;=\; \Im m\left(T^{-1}\right)\,\Re e\,(\boldsymbol{Z}) \;+\; \Re e\,(\boldsymbol{Z}) \;=\; -\,\Re e\,(\boldsymbol{Z}) \;+\; \Re e\,(\boldsymbol{Z}) \;=\; 0\quad . \tag{6}$$

Hence, $\boldsymbol{Q}$ is a real vector, in agreement with the proof given in Ref. [6] and the simple demonstration, for the 2×2 case, in Ref. [1]. As a consequence, it appears that $\boldsymbol{P}$ can be parametrised with a real linear combination of T-matrix elements (Eq. 4), to be contrasted with relation (1), derived in Ref. [27], where complex coefficients were found.

Before arguing why the result (1) for relating $\boldsymbol{P}$ and T should be preferred to expression (4), we shall first elaborate on a specific example, namely the coupled $\pi K{+}\eta K{+}\eta' K$ system.

Within the approach of Eq. (1), we define

$$\left(\begin{array}{c} P_{\pi K}\\[2mm] P_{\eta K}\\[2mm] P_{\eta' K}\end{array}\right) \;=\; \left(\begin{array}{c} \Re e\,(Z_{\pi K}) \,+\, T_{\pi K\leftrightarrow\pi K}\,iZ_{\pi K} \;+\; T_{\pi K\leftrightarrow\eta K}\,iZ_{\eta K} \;+\; T_{\pi K\leftrightarrow\eta' K}\,iZ_{\eta' K}\\[2mm] \Re e\,(Z_{\eta K}) \,+\, T_{\eta K\leftrightarrow\pi K}\,iZ_{\pi K} \;+\; T_{\eta K\leftrightarrow\eta K}\,iZ_{\eta K} \;+\; T_{\eta K\leftrightarrow\eta' K}\,iZ_{\eta' K}\\[2mm] \Re e\,(Z_{\eta' K}) \,+\, T_{\eta' K\leftrightarrow\pi K}\,iZ_{\pi K} \;+\; T_{\eta' K\leftrightarrow\eta K}\,iZ_{\eta K} \;+\; T_{\eta' K\leftrightarrow\eta' K}\,iZ_{\eta' K}\end{array}\right)\quad . \tag{7}$$

Using Eq. (3) and the definitions in Eq. (7), we obtain, after some straightforward algebra, for the relation put forward by M. R. Pennington and D. J. Wilson (Eq. (4))

$$P_{\pi K} \;=$$

$$T_{\pi K\leftrightarrow\pi K}\left[iZ_{\pi K} \;+\; \frac{1}{\det(T)}\left\{T_{\pi K\leftrightarrow\eta' K}\,T_{\eta' K\leftrightarrow\eta K} \;-\; T_{\pi K\leftrightarrow\eta K}\,T_{\eta' K\leftrightarrow\eta' K}\right\}\Re e\,(Z_{\pi K}) \;+\right.$$

$$+\frac{1}{\det(T)}\left\{T_{\pi K\leftrightarrow\eta' K}\,T_{\eta' K\leftrightarrow\eta K}\;-\;T_{\pi K\leftrightarrow\eta K}\,T_{\eta' K\leftrightarrow\eta' K}\right\}\Re e\left(Z_{\eta K}\right)\;+$$

$$\left.+\frac{1}{\det(T)}\left\{T_{\pi K\leftrightarrow\eta K}\,T_{\eta K\leftrightarrow\eta' K}\;-\;T_{\eta K\leftrightarrow\eta K}\,T_{\pi K\leftrightarrow\eta' K}\right\}\Re e\left(Z_{\eta' K}\right)\right]\;+$$

$$+\,T_{\pi K\leftrightarrow\eta K}\left[iZ_{\eta K}\;+\;\frac{1}{\det(T)}\left\{T_{\eta' K\leftrightarrow\pi K}\,T_{\eta K\leftrightarrow\eta' K}\;-\;T_{\eta K\leftrightarrow\pi K}\,T_{\eta' K\leftrightarrow\eta' K}\right\}\Re e\left(Z_{\pi K}\right)\;+\right.$$

$$+\frac{1}{\det(T)}\left\{T_{\pi K\leftrightarrow\pi K}\,T_{\eta' K\leftrightarrow\eta' K}\;-\;T_{\pi K\leftrightarrow\eta' K}\,T_{\eta' K\leftrightarrow\pi K}\right\}\Re e\left(Z_{\eta K}\right)\;+$$

$$\left.+\frac{1}{\det(T)}\left\{T_{\pi K\leftrightarrow\eta' K}\,T_{\eta K\leftrightarrow\pi K}\;-\;T_{\pi K\leftrightarrow\pi K}\,T_{\eta K\leftrightarrow\eta' K}\right\}\Re e\left(Z_{\eta' K}\right)\right]\;+$$

$$+\,T_{\pi K\leftrightarrow\eta' K}\left[iZ_{\eta K}\;+\;\frac{1}{\det(T)}\left\{T_{\eta K\leftrightarrow\pi K}\,T_{\eta' K\leftrightarrow\eta K}\;-\;T_{\eta' K\leftrightarrow\pi K}\,T_{\eta K\leftrightarrow\eta K}\right\}\Re e\left(Z_{\pi K}\right)\;+\right.$$

$$+\frac{1}{\det(T)}\left\{T_{\pi K\leftrightarrow\eta K}\,T_{\eta' K\leftrightarrow\pi K}\;-\;T_{\pi K\leftrightarrow\pi K}\,T_{\eta' K\leftrightarrow\eta K}\right\}\Re e\left(Z_{\eta K}\right)\;+$$

$$\left.+\frac{1}{\det(T)}\left\{T_{\pi K\leftrightarrow\pi K}\,T_{\eta K\leftrightarrow\eta K}\;-\;T_{\eta K\leftrightarrow\pi K}\,T_{\pi K\leftrightarrow\eta K}\right\}\Re e\left(Z_{\eta' K}\right)\right]\quad, \tag{8}$$

and similarly for $P_{\eta K}$ and $P_{\eta' K}$. Hence, by the use of Eq. (4), we find

$$\begin{aligned} Q_{\pi K} \;=\; iZ_{\pi K}\;+\frac{1}{\det(T)}\,\big[\,&\left\{T_{\pi K\leftrightarrow\eta' K}\,T_{\eta' K\leftrightarrow\eta K}\;-\;T_{\pi K\leftrightarrow\eta K}\,T_{\eta' K\leftrightarrow\eta' K}\right\}\Re e\left(Z_{\pi K}\right)\;+\\ &+\;\left\{T_{\pi K\leftrightarrow\eta' K}\,T_{\eta' K\leftrightarrow\eta K}\;-\;T_{\pi K\leftrightarrow\eta K}\,T_{\eta' K\leftrightarrow\eta' K}\right\}\Re e\left(Z_{\eta K}\right)\;+\\ &+\;\left\{T_{\pi K\leftrightarrow\eta K}\,T_{\eta K\leftrightarrow\eta' K}\;-\;T_{\eta K\leftrightarrow\eta K}\,T_{\pi K\leftrightarrow\eta' K}\right\}\Re e\left(Z_{\eta' K}\right)\,\big]\,, \end{aligned} \tag{9}$$

and similarly for $Q_{\eta K}$ and $Q_{\eta' K}$.

An immediate observation, by comparison of Eq. (7) with Eqs. (8)(9), is that the coefficients Q_i in the latter 2 equations contain elements of the T matrix, in contrast with the entirely kinematical coefficients Z_i in Eq. (7), which evidently carry the kinematical scattering cuts. The separation of the two-body dynamics from its kinematics is extremely useful for data analysis, whereas coefficients — to be fitted in practical production applications — that mix dynamics and kinematics do not seem to have much predictive power in analysing such processes. Moreover, there is another advantage in using our result (1). Namely, the complex coefficients are not only just kinematical but even completely known functions of the two-body CM energy (see Appendix A), leaving any fitting freedom limited to complex couplings. The power of this approach has already been demonstrated [30] in the simple one-channel case, applied to production processes involving the light scalar mesons $f_0(600)$ (alias σ) and $K_0^*(800)$ (alias κ), allowing to dispense with any background contributions.

Finally, there is yet another difficulty with the, in principle, simple proof of Ref. [1]. In our work on the coupling of confined systems to scattering channels [31], it was shown that the number of degrees of freedom of the T-matrix is equal to the number of confined channels,

which is usually much smaller than the number of coupled scattering channels. A very successful comparison with experiment, based on that observation, was made in Figs. 6 and 7 of Ref. [31], viz. for the coupled $\pi K + \eta K + \eta' K$ system, showing that there essentially is only one independent eigenphase. As a consequence, the T-matrix is singular. Hence, relation (3) cannot be applied in such cases. In our expressions, we carefully avoid the use of T^{-1}, thus ending up with complex coefficients for the relation between the two-body production subamplitude $\boldsymbol{P}$ and the scattering amplitude T, which nevertheless satisfies the unitarity relation (2).

Summarising, in Ref. [27] we have employed a microscopic quark-meson model, successful in meson spectroscopy and non-exotic meson-meson scattering, to derive a simple relation between production and scattering amplitudes. Although this relation involves an inhomogeneous real term as well as complex coeffients, all these are purely kinematical, known functions of the two-body energy. Moreover, the generally accepted unitarity relation (2) between production and scattering is manifestly satisfied.

In Ref. [1], M. R. Pennington and D. J. Wilson criticised our relation by showing, in the 2×2 case, how one can, in principle, rewrite it so as to end up with the usual real relation, with no inhomogeneous term. In the foregoing, we have explained why the Pennington-Wilson arguments are flawed, for two reasons. First of all, their construction to rewrite our relation for the 2×2 case, which merely amounts to solving two linearly independent real algebraic equations with two unknowns, involves the inverse of the T-matrix. Now, the latter is generally singular when coupling one or a few $q\bar{q}$ channels to several meson-meson channels, which is compatible with experimental data on $I = 1/2$ S-wave $K\pi$ scattering.

But even if it were possible to define T^{-1}, the Pennington-Wilson construction would lead to coeffcients that, albeit real, depend on a combination of T-matrix elements, thus mixing dynamics into coefficients usually adjusted freely in production analyses. Similar critiques on M. R. Pennington's approach were already formulated by the Ishidas [32].

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A Precise definition of $Z_k(E)$

In Ref. [27] we discussed the partial-wave expansion of the amplitudes for two-meson production — together with a spectator particle — and scattering, assuming $q\bar{q}$ pair creation. Hence, the coefficients bear reference to the partial wave ℓ and the flavor content α of the quark pair. We obtained [27] the following relation between production and scattering partial-wave amplitudes:

$$P^{(\ell)}_{\alpha i} \; = \; g_{\alpha i}\, j_\ell\left(p_i r_0\right) \, + \, i \sum_\nu g_{\alpha\nu}\, h^{(1)}_\ell\left(p_\nu r_0\right) \, T^{(\ell)}_{i\nu} \quad , \tag{10}$$

Accordingly, we must define

$$Z^{(\ell)}_{\alpha k}(E) \; = \; g_{\alpha k}\, h^{(1)}_\ell\left(p_k r_0\right) \quad . \tag{11}$$

In the latter equations, j_ℓ and $h^{(1)}_\ell$ stand for the spherical Bessel function and Hankel function of the first kind, respectively. These are smooth functions of the total CM energy, just like μ_k and p_k, which are the reduced mass and relative linear momentum of the two-meson system in

the k-th channel, respectively. The constants $g_{\alpha k}$ stand for the intensities of the $q\bar{q} \rightarrow MM$ couplings. A distance scale $\sim$0.6 fm (for light quarks) is represented by r_0. In the text we have stripped Z of a reference to ℓ and α.

Note, moreover, as can be easily seen from expressions (1) and (10), that the pole structures of the production and scattering amplitudes are identical, since $\Re e\,(Z_k)$, which is proportional to the spherical Bessel function in Eq. (10), is a smooth function of the total invariant mass.

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